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DIRECTED ANGLES AND INVERSION, WITH A PROOF OF SCHOUTE'S THEOREM.

By ROGER A. JOHNSON, Western Reserve University.

In this paper it is proposed to apply the method of directed angles[1] to the theory of inversion, especially to the use of the latter in studying properties of the triangle. None of the theorems below is new, but in most cases the form of presentation is new, and this form seems to be more definite and more easily workable than the usual statements.

We begin with a group of fundamental angle relations connected with the transformation of inversion; these are then applied to the triangle, yielding various interesting theorems. Thirdly, we state some of the best known properties of the so-called circles of Apollonius, and the circles of Schoute which are orthogonal to them, and finally we have a simple proof of a noted theorem of Schoute. This proof resembles to some extent a proof given by W. Gallatly,[2] but avoids some complexities. The original proof given by Schoute[3] was algebraic.

We recall the most useful properties of directed angles from the earlier paper:[4]

(*A*) The addition of directed angles follows the same rules as that of directed segments of a line. A useful identity is

$$\measuredangle l_1, l_2 + \measuredangle l_3, l_4 = \measuredangle l_1, l_4 + \measuredangle l_3, l_2.$$

(*B*) Three points A, B, C are collinear if and only if

$$\measuredangle ACB = 0$$

or, in other words, if for any other point D

$$\measuredangle ACD = \measuredangle BCD.$$

(*C*) Four points A, B, C, D lie on a circle or line if and only if

$$\measuredangle ABC = \measuredangle ADC.$$

(*D*) The Miquel formula for the angles of the pedal triangle of a point P with regard to triangle $A_1A_2A_3$. If PP_1, PP_2, PP_3 are the $\perp$s to the sides, then

$$\measuredangle A_2PA_3 = \measuredangle P_2P_1P_3 + \measuredangle A_2A_1A_3, \text{ etc.}$$

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[1] See a paper by R. A. Johnson, *Directed Angles in Elementary Geometry*, this MONTHLY, Vol. XXIV, page 101, 1917.

[2] *The Modern Geometry of the Triangle*, Chap. VIII.

[3] Schoute, *Over een neawer verband tusschen hoek en cirkel van Brocard*, Amsterdam Acad. Proceedings, Series III, vol. 3, 1887, page 22; see Coolidge, *Geometry of the Circle and the Sphere*, page 127.

[4] One who desires to extract the essence of the present note without using the method of directed angles may proceed by drawing in each case a figure, and interpreting any statement of the form $\measuredangle x = \measuredangle y$ as meaning that the angles x and y of the figure are either equal or supplementary or differ by 180°.

Consider now an inversion with regard to a fixed circle with center O, radius r.

THEOREM 1. *Two pairs of inverse points are either collinear with O, or concyclic on a circle orthogonal to the circle of inversion.*

For

$$\overline{OP}\cdot\overline{OP'}=\overline{OQ}\cdot\overline{OQ'}=r^2.$$

THEOREM 2. *Triangles OPQ, $OQ'P'$ are inversely similar.*

For angle O is common; and by (C) and theorem 1 we have

$$\measuredangle OPQ=\measuredangle P'Q'O,\qquad \measuredangle PQO=\measuredangle OP'Q'.$$

COROLLARY. *The distance between two points, and the distance between their inverses, are connected by*

$$\overline{P'Q'}=\overline{QP}\,\frac{r^2}{\overline{OP}\cdot\overline{OQ}}.$$

THEOREM 3. *If P, Q, R are any three points, P', Q', R' their inverses,*

$$\measuredangle QPR+\measuredangle Q'P'R'=\measuredangle QOR.$$

Proof.

$$\measuredangle QPO=\measuredangle OQ'P',$$

$$\measuredangle OPR=\measuredangle P'R'O.$$

Adding,

$$\measuredangle QP,PO+\measuredangle PO,PR=\measuredangle OQ',Q'P'+\measuredangle Q'R',R'O,$$

or, applying (A),

$$\measuredangle QPR=\measuredangle R'P'Q'+\measuredangle Q'OR',$$

$$\measuredangle QPR+\measuredangle Q'P'R'=\measuredangle QOR.$$

This formula is exceedingly powerful; in fact, it may be ranked in point of importance with the Miquel formula (D), which it resembles in form. From it many of the important properties of inversion are immediate corollaries.

COROLLARY 1. *For any four points and their inverses,*

$$\measuredangle PQR+\measuredangle RSP=\measuredangle R'Q'P'+\measuredangle P'S'R'.$$

COROLLARY 2. *The inverse of a circle not passing through the center of inversion is also a circle.*

COROLLARY 3. *The inverse of a straight line not through the center of inversion is a circle through that point, and conversely.*

COROLLARY 4. *The angle of intersection of two circles is the same as that of the inverse circles. This is true also for any intersecting curves.*

We turn now to the triangle.

THEOREM 4. *If two points are mutually inverse with regard to the circumcircle of a triangle $A_1A_2A_3$, their pedal triangles are inversely similar.*

For each vertex of the triangle is self-inverse, hence we have from Theorem 3,

$$\measuredangle A_2PA_3+\measuredangle A_2P'A_3=\measuredangle A_2OA_3.$$

Applying (D),

$$\measuredangle A_2A_1A_3+\measuredangle P_2P_1P_3+\measuredangle A_2A_1A_3+\measuredangle P_2'P_1'P_3'=\measuredangle A_2OA_3.$$

But

$$2\measuredangle A_2A_1A_3=\measuredangle A_2OA_3;$$

hence

$$\measuredangle P_2P_1P_3=\measuredangle P_3'P_1'P_2'.$$

Since all the angles of the pedal triangles satisfy such equations, the triangles are inversely similar.

We may regard as a corollary to this theorem the fact (otherwise much more easily proved) that the pedal triangle of a point on the circumcircle reduces to a line.

THEOREM 5. *If an inversion is performed with any center P, the inverses B_1, B_2, B_3 of any three points A_1, A_2, A_3 form a triangle similar to the pedal triangle of P with regard to $\triangle A_1A_2A_3$.*

For since

$$\measuredangle A_2PA_3 = \measuredangle A_2A_1A_3 + \measuredangle B_2B_1B_3 \qquad \text{(Theorem 3)}$$

$$\measuredangle A_2PA_3 = \measuredangle A_2A_1A_3 + \measuredangle P_2P_1P_3 \qquad (D)$$

it follows that

$$\measuredangle P_2P_1P_3 = \measuredangle B_2B_1B_3, \quad \text{etc.}$$

COROLLARY. *If a triangle is subjected to an inversion with regard to two points which are inverse with regard to the circumcircle, the resulting triangles are inversely similar.*

THEOREM 6. *There is a single point which may be taken as center of inversion, in order that a given triangle may be transformed into a triangle similar in a given sense to a given triangle.*

For the center of inversion P is located uniquely by the equations

$$\measuredangle A_2PA_3 = \measuredangle A_2A_1A_3 + \measuredangle B_2B_1B_3,$$

$$\measuredangle A_3PA_1 = \measuredangle A_3A_2A_1 + \measuredangle B_3B_2B_1.$$

An exceptional case arises when the given triangles are inversely similar.

THEOREM 7. *If four points are subjected to an inversion, the pedal triangle of one of the points with regard to the other three is inversely similar to the corresponding pedal triangle for the inverted points.*

Let an inversion carry A_1, A_2, A_3, P into B_1, B_2, B_3, Q respectively; and let the pedal triangle of P with regard to $A_1A_2A_3$ be $P_1P_2P_3$, and that of Q with regard to $B_1B_2B_3$ be $Q_1Q_2Q_3$. Then

$$\measuredangle A_2PA_3 + \measuredangle B_2QB_3 = \measuredangle A_2OA_3,$$

$$\measuredangle A_2A_1A_3 + \measuredangle P_2P_1P_3 + \measuredangle B_2B_1B_3 + \measuredangle Q_2Q_1Q_3 = \measuredangle A_2OA_3.$$

But

$$\measuredangle A_2A_1A_3 + \measuredangle B_2B_1B_3 = \measuredangle A_2OA_3$$

so that

$$\measuredangle P_2P_1P_3 + \measuredangle Q_2Q_1Q_3 = 0,$$

as was to be proved.

We pass now to the circles of Apollonius and the isodynamic points. The locus of a point whose distances from two vertices of a triangle are proportional to the adjacent sides, is a circle through the third vertex. In any triangle there are three such circles, called the circles of Apollonius. They are well known, and we shall merely mention a few of their properties. It is easily proved that the center of the circle through A_1 lies on A_2A_3; that the circle passes through the feet of the bisectors of angle A_1; and that the tangent to the circumcircle at A_1 passes through its center. The three circles intersect at two points, called isodynamic points; the distances of these from the sides of the triangle are inversely as the lengths of these sides.

THEOREM 8. *If two points are mutually inverse with regard to the circle of*

Apollonius C_1 through A_1, their pedal triangles are inversely similar in the sense

$$\triangle P_1P_2P_3 \eqsim \triangle Q_1Q_3Q_2.$$

First, A_2 and A_3 are inverse with regard to this circle, for if L_1 is its center, $\overline{L_1A_2}\cdot\overline{L_1A_3} = \overline{L_1A_1}^2$, since L_1A_1 is tangent to the circumcircle. Hence we have two pairs of inverse points, A_2, A_3 and P, Q, and further A_1 is inverse to itself

$$\measuredangle A_1PA_2 + \measuredangle A_1QA_3 = \measuredangle A_1L_1A_3.$$

Applying (D),

$$\measuredangle A_1A_3A_2 + \measuredangle P_1P_3P_2 + \measuredangle A_1A_2A_3 + \measuredangle Q_1Q_2Q_3 = \measuredangle A_1A_3A_2 + \measuredangle A_1A_2A_3,$$

whence

$$\measuredangle P_1P_3P_2 = \measuredangle Q_3Q_2Q_1.$$

Similarly,

$$\measuredangle P_3P_2P_1 = \measuredangle Q_1Q_3Q_2.$$

COROLLARY. *If P lies on the circle C_1, its pedal triangle is isosceles in the sense $\overline{P_1P_2} = \overline{P_1P_3}$. The pedal triangle of each isodynamic point is equilateral. It follows that the isodynamic points are inverse with regard to the circumcircle.*

This last result may also be deduced from the fact that these points are the intersections of circles orthogonal to the circumcircle.

Also orthogonal to the circles of Apollonius is their line of centers. This may easily be shown to be the polar, with regard to the circumcircle, of the symmedian point K; therefore the inverse of this line, which is the Brocard circle, is also orthogonal to the circles of Apollonius. We may designate the coaxial system which includes the circumcircle, the Brocard circle, the Lemoine line (polar of the symmedian point) and the isodynamic points, as the Schoute system of circles. Its members are orthogonal to the circles of Apollonius.

THEOREM 9. *If an inversion is performed with regard to a circle about either isodynamic point, the given triangle transforms into an equilateral triangle whose center is at the inverse of the other isodynamic point, the circles of Apollonius invert into the altitudes of this triangle, and the circles of Schoute into circles concentric with it.*

THEOREM 10. *In an equilateral triangle, the locus of a point whose pedal triangle has a constant Brocard angle is a circle concentric with the triangle.*

For let $B_1B_2B_3$ be an equilateral triangle whose side is b; let O denote its center; let $P_1P_2P_3$ be the pedal triangle of any point P.

We have

$$\overline{P_2P_3} = \overline{B_1P}\sin B_1 = \overline{B_1P}\,\frac{\sqrt{3}}{2}$$

$$\overline{P_2P_3}^2 + \overline{P_3P_1}^2 + \overline{P_1P_2}^2 = \tfrac{3}{4}(\overline{B_1P}^2 + \overline{B_2P}^2 + \overline{B_3P}^2).$$

But by a well known theorem,

$$\overline{B_1P}^2 + \overline{B_2P}^2 + \overline{B_3P}^2 = (\overline{B_1O}^2 + \overline{B_2O}^2 + \overline{B_3O}^2) + 3\overline{OP}^2 = b^2 + 3\overline{OP}^2$$

so that finally

$$\overline{P_2P_3}^2 + \overline{P_3P_1}^2 + \overline{P_1P_2}^2 = \tfrac{3}{4}b^2 + \tfrac{9}{4}\overline{OP}^2$$

and we see that if P moves on a circle about O, the sum of the squares of the sides of its pedal triangle is constant. Also it is well known that in the same case, the area of the pedal triangle is constant, and since

$$\cot\omega = \frac{\overline{P_2P_3}^2 + \overline{P_3P_1}^2 + \overline{P_1P_2}^2}{4\Delta}$$

the theorem is proved.

THEOREM 11. (SCHOUTE'S theorem) *In any triangle, the locus of a point whose pedal triangle has a constant Brocard angle is a circle coaxial with the circumcircle and the Brocard circle.*

The proof is effected by performing the inversion of theorem 9, noting theorem 7, and applying Theorem 10.

COROLLARY. *In particular, the locus of a point whose pedal triangle has the same Brocard angle (in the same sense) as the given triangle is the Brocard circle; and for the same Brocard angle but with the triangle described in the opposite sense, the locus is the Lemoine line.*

For, in the first place, the Brocard circle contains a point O, whose pedal triangle is similar to the given triangle; and further, the points of this circle and of the Lemoine line are mutually inverse with regard to the circumcircle.

All previous proofs by elementary methods, even of the corollary, have made use of a very awkward method of vertical projections from one plane to another (cf. Gallatly, l. c., or Emmerich, Die Brocard'sche Gebilde, §§ 60, 61). It will be noted that most of the steps of the present proof may fairly be regarded as general theorems, very little special proof being required for the theorem itself.

A SIMPLE RELATION BETWEEN ELEMENTARY NUMBER-THEORY AND ELEMENTARY PROJECTIVE GEOMETRY.

By AUBREY J. KEMPNER, Urbana, Ill.

In *L'Enseignement Mathématique,* 1916, p. 332, Mr. A. Reymond has published a note in which, with slight modifications, the following simple graphical method of determining whether a given integer is a prime number, is explained.

Write in a vertical and in a horizontal row the numbers 0, 1, 2, 3, 4, $\cdots$, as far as one likes, and fill out Fig. 1 in the way indicated. Evidently every number in our first vertical row, for example 6, will have for factors just the numbers which are in the same horizontal row, in this case 1, 2, 3, 6. Here the number 1 and the number 6 are both considered factors of 6. The number 7 has only the

0	1	2	3	4	5	6	7	8	9	10
1	1									
2	1	2								
3	1		3							
4	1	2		4						
5	1				5					
6	1	2	3			6				
7	1						7			
8	1	2		4				8		
9	1		3						9	
10	1	2			5					10

FIG. 1.

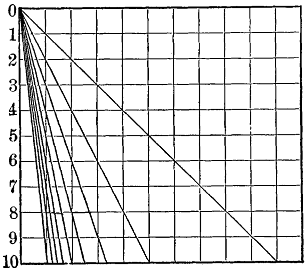

FIG. 2.